Ergodic Theory and Measured Group Theory
Lecture 26

Before continuing cost calculations, let's discuss measure equivalence of groups, which is to measured group theory the same as quasi-isometry is to geometric group theory, and this strong analogy makes these two subjects siblings, with two-way applications in each. Here is the motivating statement:

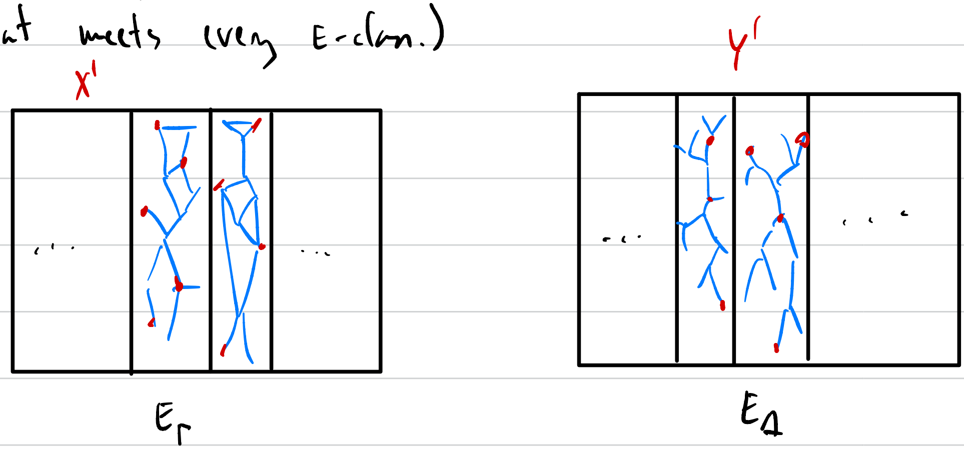
Prop (Gromov). Ctbl groups Γ and Δ are quasi-isometric $\Leftrightarrow$ they admit a topological coupling, i.e. they admit continuous actions $\Gamma \curvearrowright X \curvearrowleft \Delta$ on a locally compact space X s.t.

(i) the action of Γ and Δ commute

(ii) these actions are cocompact, i.e. admit a compact transversal Y s.t. $\bigsqcup_{\gamma \in \Gamma} \gamma \cdot Y = X = \bigsqcup_{\delta \in \Delta} \delta \cdot Y$, in particular, the action is free, and proper ($\forall$ compact K, $\# \gamma \in \Gamma$ s.t. $\gamma K \cap K \neq \emptyset$ is finite, same for Δ).

Def. Pmp actions $\Gamma \curvearrowright (X, \mu)$ and $\Delta \curvearrowright (Y, \nu)$ are called stably

orbit equivalent if $E_\Gamma|_{X'}$ is measure isomorphic to $E_\Delta|_{Y'}$, where X', Y' are Borel complete sections for E_Γ and E_Δ, respectively. (A complete section for a CBER E is a set that meets every E-class.)



This is analogous to quasi-isometry and the statement makes this more convincing. Γ

Γ, Δ are stably orbit equivalent

$\Updownarrow$

<u>Prop (Gromov, Furman).</u> Ctbl groups Γ, Δ admit stably orbit equivalent free pmp actions $\iff$ they admit a **measure coupling**, i.e. measure-preserving actions $\Gamma \curvearrowright (X, \rho) \curvearrowleft \Delta$ on a σ-finite standard measure space (X, ρ) s.t.

(i) the actions commute

(ii) they are free and admit measurable transversals of finite measure.

Def. Ctbl groups Γ and Δ are called **measure equivalent**, denoted **ME**, if they admit a measure coupling.

Thus, Γ ME Δ $\iff$ Γ, Δ are stably orbit equivalent.

Def. For a ctbl gp Γ, define $\mathrm{cost}(\Gamma) := \inf_{E_\Gamma} \mathrm{cost}(E_\Gamma)$, where E_Γ ranges over the orbit eq. rel. of all free pmp actions of Γ.

Here is R. Tucker-Drob's table of our current knowledge about ME-classes:

ME-class	Cost description	Group-theoretic description
Class of $\{e\}$	Treeable, $\mathrm{cost}(\Gamma)<1$	Finite groups
Class of $\mathbb{F}_1 = \mathbb{Z}$	Treeable, $\mathrm{cost}(\Gamma)=1$	Infinite amenable groups
Class of $\mathbb{F}_2$	Treeable, $\mathrm{cost}(\Gamma)<\infty$	???
Class of $\mathbb{F}_\infty$	Treeable, $\mathrm{cost}(\Gamma)=\infty$	???

We proved II column $\iff$ III column. Proof of I $\iff$ III is an exercise.

We sketched the proof of I $\iff$ II. I $\iff$ III is the Ornstein-Weiss thm.

Open problems

Back to cost computations.

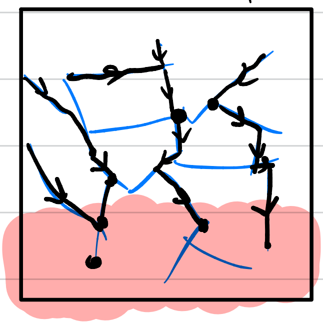
We saw that if a pmp CBER is finite, then its cost is $= 1 - \mu(\text{transversal}) < 1$. What about the aperiodic CBERs, i.e. those with only ∞ classes?

Prop. If a pmp CBER E is aperiodic (hence nowhere smooth), then its cost is ≥ 1.

Proof. Let G be any graphing of E and let M be a complete section of $< \varepsilon$ measure (which exists by the marker lemma).

(E-class

$\vec{H}$



M

We build an acyclic subgraph $H \subseteq G$ with a Borel directing $\vec{H}$ as follows: direct every point in $X \setminus M$ towards M through the lex-least shortest path. Indeed $\vec{H}$ is a Borel directing of H, i.e., if $(x,y) \in \vec{H}$ then $(y,x) \notin \vec{H}$.

Then $\mathrm{Cost}_\mu(G) \geq \mathrm{Cost}_H(H) = \int_X \mathrm{outdeg}_{\vec{H}}(x)\, d\mu(x) = \int_{X \setminus M} 1\, d\mu(x)$
$= \mu(X) - \mu(M) \geq 1 - \varepsilon$.

Since ε is arbitrary, $\mathrm{Cost}_\mu(G) \geq 1$. $\square$

Which eq. rel. have cost 1 and which achieve it?

Def. An eq. rel. E on a st. Borel space X is called *hyperfinite* if E is an increasing union of finite Borel eq. rel., i.e. $E = \bigcup\limits^{\uparrow}_n E_n$, each E_n finite Borel.

Typical example is $\mathbb{E}_0$, indeed $\mathbb{E}_0 = \bigcup\limits^{\uparrow}_n E_n$, here for

$x,y \in 2^{\mathbb{N}}$, $\quad x E_n y :\Leftrightarrow \forall m \geq n \quad x(m) = y(m)$.

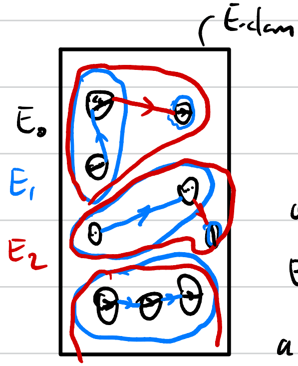
Theorem (Dougherty–Jackson–Kechris). For a CBER E, TFAE:

(1) E is hyperfinite.

(2) E is is induced by a Borel action of $\mathbb{Z}$.

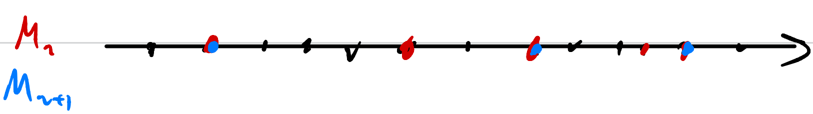
(3) $E \leq_B \mathbb{E}_0$.

Proof-sketch. (1) $\Rightarrow$ (2).



Let $<$ be a linear ordering on X. Let this be the ordering of points in each E_0-class. Then we create a directed line on each E_1-class by adding edges between the little lines created for each E_0-class. Continuing this way we create a graphing consisting of directed biinfinite lines, modulo a smooth Borel set.

(2) $\Rightarrow$ (1). Take a vanishing sequence of marker sets (M_n). Look at one infinite E-class: the E_n-classes are



the intervals between two consecutive M_n points. E_{n+1}-classes are the intervals between ... M_{n+1}-points.

(3)$\Rightarrow$(1). Hyperfiniteness is closed downward: just pull-back the witnesses to hyperfiniteness.

(1)$\Rightarrow$(3). If $E = \bigcup\!\!\!\uparrow_n E_n$, E_n finite, then $\forall x, y \in X$, $x E y \iff \forall^\infty n\ [x]_{E_n} = [y]_{E_n}$. Encoding the finite classes $[x]_{E_n}$ into binary sequences creates the reduction to E_0. $\square$

<u>Theorem (Levitt)</u>. Let E be a pmp hyperfinite aperiodic CBER. Then $\mathrm{cost}(E)=1$, which is achieved by any treeing of E.

Proof. Because E is induced by a $\mathbb{Z}$-action, $\mathrm{cost}(E) \le 1$, and since it's aperiodic $\mathrm{cost}(E) \ge 1$, so $\mathrm{cost}(E) = 1$. Now let T be any treeing of E and let $E = \bigcup\!\!\!\uparrow_n E_n$, where each E_n is finite. $\forall x \in X$, $\deg_T(x) = \lim\limits_{n\to\infty} \deg_{T_n}(x)$, where $T_n := T \cap E_n$, so $\mathrm{cost}(T) = \frac{1}{2} \int_X \deg_T(x)\, d\mu(x) =$

(by the MCT) $= \frac{1}{2} \lim\limits_{n\to\infty} \int_X \deg_{T_n}(x)\, d\mu(x) = \lim\limits_{n\to\infty} \mathrm{cost}(E_{T_n}) \le 1$, hence $\mathrm{cost}(T) = 1$. $\square$

Let's deduce that there are nonhyperfinite eq. rel.

Prop. Let $E_{\mathbb{F}_2}$ be the orb. eq. rel. of a a.e. free pmp action of $\mathbb{F}_2$, e.g. Bernoulli shift $\mathbb{F}_2 \curvearrowright \left(2^{\mathbb{F}_2}, \{\frac{1}{2}, \frac{1}{2}\}^{\mathbb{F}_2}\right)$. Then $E_{\mathbb{F}_2}$ is nowhere hyperfinite.

Proof. The Schreier graph of this action is a 4-regular treeing, so its cost is 2, while we just saw that any treeing of a hyperfinite eq. rel. has cost ≤ 1. $\square$

The Borel reducibility hierarchy of CBERs: